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Geometric Approach to Three-Dimensional Missile Guidance Problem

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The Frenet–Serret formula in classical differential geometry curve theory and the characteristics of a fictitious missile pointing velocity vector are used to design missile guidance curvature command. Qualitative analysis is conducted to study capture capability of the designed guidance command in three-dimensional engagements. The region that miss can occur is derived in terms of the tangential component of the kinematics equation. Then, sufficient initial condition is derived, which, with target's maneuvering information, can guarantee capture for arbitrary target maneuver. To validate this capture conclusion, two simple missile torsion commands are introduced to rotate direction of the missile curvature command to ensure that the curvature command formula is well defined. Such missile roll strategy may also help to improve capture capability in the final phase of an engagement.

Nomenclature

A = missile guidance curvature command proportional constant
$\boldsymbol{b}$ = unit binormal vector
$\boldsymbol{e}$ = unit vector with subscript
k = curvature or curvature command response
m = velocity ratio, (V_t/V_m)
$\boldsymbol{n}$ = unit normal vector
$\boldsymbol{r}$ = line-of-sight (LOS) vector or position vector with subscript
s = arc length along missile trajectory
$\boldsymbol{t}$ = unit tangent vector
$\boldsymbol{V}$ = velocity vector
τ = torsion or torsion command response
$\boldsymbol{\omega}$ = LOS rate vector

Subscripts

m = missile
mc = missile guidance command input
me = fictitious missile missing velocity direction or at miss condition
mp = fictitious missile pointing velocity direction
r = along LOS vector
t = target
te = measured or estimated target curvature command
ω = normal to LOS vector
0 = initial condition

Superscript

$'$ = spatial derivative with respect to s

I. Introduction

MOST analytic studies of the missile guidance problem in two-dimensional engagements are based on the assumption that the missile follows the proportional guidance command. The researchers then try to solve a system of coupled nonlinear ordinary differential equations or to apply an optimal control method to design different guidance commands, such as in Refs. 1–12. In three-dimensional engagements, Adler[13] applies the classical differential geometry surface theory to study the missile guidance problem and derives proportional navigation law in terms of geometry curvatures. Lin[14] applies linear quadratic Gaussian theory to design missile acceleration commands and then transforms these commands into normal acceleration and bank attitude commands for bank-to-turn missile maneuver. Cochran et al.[15] study true proportional navigation guidance problems. The closed-form solution is derived in the form of an elliptic function for a nonmaneuvering target. Kumar et al.[16] apply the optimal control technique to design three-phase feedback guidance laws. Lin and Lee[17] apply the linear exponential quadratic Gaussian method to derive the optimal missile pitch acceleration and roll rate commands with Poisson square wave target maneuvering model. Yang and Yang[18] study realistic true proportional navigation (RTPN) guidance problems. The closed-form solutions are derived for missile following the RTPN guidance command with a nonmaneuvering target and a maneuvering target that follows ideal proportional navigation command. Also, in Ref. 19, Yang and Yang derive the analytic solutions for a missile following the generalized proportional navigation guidance command in terms of elliptic functions. In the literature listed for three-dimensional engagements, bank-to-turn guidance commands are designed in Refs. 14, 16, and 17 and skid-to-turn guidance commands are designed in Refs. 15, 18, and 19.

Basically, some specific target maneuvers are assumed in all of the literature listed. For example, in two-dimensional engagements, Murtaugh and Criel[1] assumes a nonmaneuvering target, Guelman[2–4] assumes nonmaneuvering and constant maneuvering targets, Yuan and Chern[7,8] assume zero and specific target accelerations, and Yang and Yeh[9,10] assume zero and constant target accelerations. Most of the studies in three-dimensional engagements assume constant target maneuvers.[13–16] In Ref. 17, a model for target escape maneuver is assumed with a specific acceleration command distribution. Yang and Yang[18,19] assume targets follow either ideal proportional navigation or zero acceleration. These assumptions on target maneuvers usually are not satisfied in practical applications because most real world targets follow variable roll and turn maneuvers during encounters. Therefore, the capture of a variable maneuvering target is an important practical issue in missile guidance that has not been carefully discussed. In this paper, this issue is addressed by designing a guidance law that, with a clearly defined sufficient initial condition and sufficient target motion information, can guarantee capture for arbitrary target escape maneuver. This should help to outline the benefits of being able to provide better target motion information to a missile's guidance system.

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In three-dimensional engagements, trajectories of the missile and target belong to three-dimensional space curves. Not only do the instantaneous osculating planes of the missile and target differ and vary continuously, but also all of the angular velocity vectors change both in magnitude and direction as the engagement proceeds. The trajectories, in this case, are determined by the curvature and torsion commands applied in directions normal to and along the unit tangent vectors of the missile and target, respectively. Referring to Refs. 20 and 21, both trajectories of the missile and target belong to space curves. Classical differential geometry curve theory is a study of the characteristics of space curves and has their characteristics formulated in terms of the curvature and torsion parameters. Therefore, it may give a different approach to the study of atmospheric trajectory planning problem.

The curve theory in classical differential geometry and the concept of a fictitious missile pointing velocity vector will be applied to derive missile guidance curvature command. Then, capture capability of this command in three-dimensional engagements will be qualitatively evaluated by studying spatial variation of the magnitude of the tangential component of the kinematics equation. A sufficient initial launch envelope is determined, which, with target's maneuvering information, can guarantee capture for an arbitrary target maneuver. The torsion command will be selected to ensure that the designed missile guidance curvature command is well defined.

II. Problem Formulation and Derivation of Missile Guidance Commands

In this section, the Frenet formula[20] and the fictitious missile pointing velocity concept will be applied to derive the missile curvature command. Then, capture capability of this command in three-dimensional engagements will be qualitatively studied. The missile torsion command will be derived based on a different requirement, i.e., the condition required to guarantee the capture result of the designed curvature command. The assumptions of point masses dynamics and constant speeds are made for both missile and target in this simplified three-dimensional missile guidance problem. The effects of these assumptions are evaluated in computer simulation case 3, later in this paper. Also, speed ratio $V_t/V_m = m = \text{const} < 1$ is assumed, which is valid if curvatures of the trajectories of missile and target were treated as command forces to change their velocity directions only.

A. Kinematics and Dynamics Equations

In this section, the basic kinematics and dynamics equations are derived. Based on Fig. 1, we have

$$\boldsymbol{r}_m = \boldsymbol{r}_t - r\boldsymbol{e}_r$$

Taking the derivative of the preceding equation with respect to s, we have the following basic kinematics equation and its scalar components along $\boldsymbol{e}_r$ and $\boldsymbol{e}_\omega \times \boldsymbol{e}_r$ directions:

$$\boldsymbol{t}_m = m\boldsymbol{t}_t - r'\boldsymbol{e}_r - r\omega(\boldsymbol{e}_\omega \times \boldsymbol{e}_r) \tag{1}$$

$$r' = (m\boldsymbol{t}_t - \boldsymbol{t}_m)\cdot\boldsymbol{e}_r \tag{2}$$

$$r\omega = (m\boldsymbol{t}_t - \boldsymbol{t}_m)\cdot\boldsymbol{e}_\omega \times \boldsymbol{e}_r \tag{3}$$

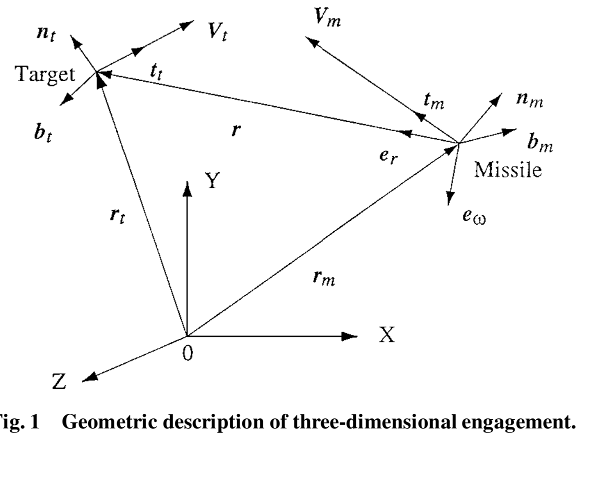

Fig. 1 Geometric description of three-dimensional engagement.

Because rotation of the line-of-sight (LOS) vector is completely determined by the components of $\boldsymbol{t}_m$ and $m\boldsymbol{t}_t$ normal to the LOS vector, the unit rotational vector of the LOS vector $\boldsymbol{e}_\omega$ is also normal to the LOS vector. That is, we have

$$\boldsymbol{e}_\omega\cdot\boldsymbol{e}_r = 0, \qquad \boldsymbol{e}'_\omega\cdot\boldsymbol{e}_\omega = 0 \tag{4}$$

Taking the derivative of Eq. (1) with respect to s and applying the Frenet formula,[20,21] we have the basic dynamics equation

$$k_m\boldsymbol{n}_m = m^2k_t\boldsymbol{n}_t - r''\boldsymbol{e}_r - 2r'\omega\boldsymbol{e}_\omega\times\boldsymbol{e}_r - r\omega'\boldsymbol{e}_\omega\times\boldsymbol{e}_r$$
$$- r\omega\boldsymbol{e}'_\omega\times\boldsymbol{e}_r - r\omega^2\boldsymbol{e}_\omega\times(\boldsymbol{e}_\omega\times\boldsymbol{e}_r) \tag{5}$$

Because

$$\boldsymbol{e}_\omega\times(\boldsymbol{e}_\omega\times\boldsymbol{e}_r) = -\boldsymbol{e}_r, \qquad (\boldsymbol{e}'_\omega\times\boldsymbol{e}_r)\cdot(\boldsymbol{e}_\omega\times\boldsymbol{e}_r) = 0 \tag{5a}$$

the scalar components of Eq. (5) along the $\boldsymbol{e}_r$ and $\boldsymbol{e}_\omega\times\boldsymbol{e}_r$ directions are

$$r'' - r\omega^2 = \left(m^2k_t\boldsymbol{n}_t - k_m\boldsymbol{n}_m\right)\cdot\boldsymbol{e}_r \tag{6}$$

$$r\omega' + 2r'\omega = \left(m^2k_t\boldsymbol{n}_t - k_m\boldsymbol{n}_m\right)\cdot(\boldsymbol{e}_\omega\times\boldsymbol{e}_r) \tag{7}$$

It is noted that $\boldsymbol{t}_m$ and $\boldsymbol{t}_t$ are along the velocity vector directions and $\boldsymbol{n}_m$ and $\boldsymbol{n}_t$ are along the normal acceleration directions of missile and target, respectively. In three-dimensional engagements, because the rotational vector of the LOS vector will change not only in magnitude but also in direction, the rate of change of the unit rotational vector $\boldsymbol{e}_\omega$ can be computed from Eq. (1) as follows:

$$r\omega\boldsymbol{e}_\omega = (\boldsymbol{t}_m - m\boldsymbol{t}_t)\times\boldsymbol{e}_r \tag{8}$$

Taking the derivative of the last equation with respect to s, we have

$$r'\omega\boldsymbol{e}_\omega + r\omega'\boldsymbol{e}_\omega + r\omega\boldsymbol{e}'_\omega = \left(k_m\boldsymbol{n}_m - m^2k_t\boldsymbol{n}_t\right)\times\boldsymbol{e}_r$$
$$+ (\boldsymbol{t}_m - m\boldsymbol{t}_t)\times(\boldsymbol{\omega}\times\boldsymbol{e}_r) \tag{9}$$

From Eqs. (1) and (9), we have

$$\boldsymbol{e}'_\omega = \frac{\left(k_m\boldsymbol{n}_m - m^2k_t\boldsymbol{n}_t\right)\times\boldsymbol{e}_r - (2r'\omega + r\omega')\boldsymbol{e}_\omega}{r\omega} \tag{10}$$

Also, from Eq. (8), the unit rotational vector of LOS vector $\boldsymbol{e}_\omega$ has the following form:

$$\boldsymbol{e}_\omega = \frac{(\boldsymbol{t}_m - m\boldsymbol{t}_t)\times\boldsymbol{e}_r}{r\omega} \tag{11}$$

In the following section, based on the basic kinematics and dynamics equations plus the just-derived geometric relations between the unit vectors, the fictitious missile pointing velocity concept will be used to derive the missile guidance curvature command.

B. Derivation of Missile Guidance Curvature Command

Assume there is a fictitious missile velocity vector pointing in a direction such that the corresponding fictitious LOS rate (LOSR) is zero. Therefore, if the missile velocity vector coincides with the direction of this fictitious pointing velocity vector, the instantaneous LOSR is zero. From Eq. (1), we have

$$\boldsymbol{t}_{mp} = m\boldsymbol{t}_t - r'_{mp}\boldsymbol{e}_r, \qquad r'_{mp} = (m\boldsymbol{t}_t - \boldsymbol{t}_{mp})\cdot\boldsymbol{e}_r \tag{12}$$

From Eq. (12), we have

$$(m\boldsymbol{t}_t - \boldsymbol{t}_{mp})\cdot(\boldsymbol{e}_\omega\times\boldsymbol{e}_r) = 0$$

Taking the derivative of the preceding equation with respect to s and applying the Frenet formula, we have

$$m^2k_t\boldsymbol{n}_t\cdot(\boldsymbol{e}_\omega\times\boldsymbol{e}_r) + m\boldsymbol{t}_t\cdot(\boldsymbol{e}'_\omega\times\boldsymbol{e}_r) + m\boldsymbol{t}_t\cdot(\boldsymbol{e}_\omega\times(\boldsymbol{\omega}\times\boldsymbol{e}_r))$$
$$- k_{mp}\boldsymbol{n}_{mp}\cdot(\boldsymbol{e}_\omega\times\boldsymbol{e}_r) - \boldsymbol{t}_{mp}\cdot(\boldsymbol{e}'_\omega\times\boldsymbol{e}_r)$$
$$- \boldsymbol{t}_{mp}\cdot(\boldsymbol{e}_\omega\times(\boldsymbol{\omega}\times\boldsymbol{e}_r)) = 0$$

From the preceding equation and Eq. (12), the curvature command of the fictitious missile pointing velocity vector (or curvature of the fictitious missile trajectory) k_{mp} is

$$k_{mp} = m^2 k_t \frac{\boldsymbol{n}_t \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)}{\boldsymbol{n}_{mp} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} + r'_{mp} \frac{\boldsymbol{e}_r \cdot (\boldsymbol{e}_\omega \times (\boldsymbol{\omega} \times \boldsymbol{e}_r))}{\boldsymbol{n}_{mp} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} \tag{13}$$

From Eq. (5a), Eq. (13) can also be expressed as

$$k_{mp} = m^2 k_t \frac{\boldsymbol{n}_t \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)}{\boldsymbol{n}_{mp} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} - \frac{r'_{mp}\omega}{\boldsymbol{n}_{mp} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} \tag{14}$$

The direction of application of k_{mp}, i.e., $\boldsymbol{n}_{mp}$, can be found by taking the derivative of Eq. (12) with respect to s:

$$k_{mp}\boldsymbol{n}_{mp} = m^2 k_t \boldsymbol{n}_t - r''_{mp}\boldsymbol{e}_r - r''_{mp}\omega(\boldsymbol{e}_\omega \times \boldsymbol{e}_r)$$

From Eq. (14), the curvature command of the fictitious missile pointing velocity vector is applied in the $\boldsymbol{n}_{mp}$ direction and has a component along $\boldsymbol{e}_\omega \times \boldsymbol{e}_r$ equal to that of the curvature command of the target times m^2 along the $\boldsymbol{n}_t$ direction. The second term of Eq. (14) just accounts for the nonzero LOSR, which happens because the actual velocity vector of the missile is not along the $\boldsymbol{t}_{mp}$ direction.

By adding a proportional constant A into the second term of the curvature command of the fictitious missile pointing velocity vector derived in Eqs. (13) and (14) and replacing the fictitious r'_{mp} with actual closing velocity r', the following missile curvature command is proposed:

$$k_m = m^2 k_t \frac{\boldsymbol{n}_t \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)}{\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} + Ar'\omega \frac{\boldsymbol{e}_r \cdot (\boldsymbol{e}_\omega \times (\boldsymbol{e}_\omega \times \boldsymbol{e}_r))}{\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} \tag{15}$$

or

$$k_m = m^2 k_t \frac{\boldsymbol{n}_t \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)}{\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} - \frac{Ar'\omega}{\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} \tag{16}$$

Substituting Eq. (15) into the dynamics relation (5), we have

$$2r'\omega + r\omega' = Ar'\omega$$

The closed-form solution of the LOSR is

$$\omega = \omega_0 (r/r_0)^{A-2} \tag{17}$$

In Eq. (17), ω and ω_0 represent the magnitude of the LOSR vector and they are always greater than or equal to zero. If $A > 2$, ω will decrease as r decrease during the engagement.

According to these results, the designed guidance law follows the concept of pure proportional navigation guidance in that the guidance command is applied in a direction normal to the velocity vector of the missile. Also, it has the same ω response as in true proportional navigation guidance with a nonmaneuvering target. But, in this case, relation (17) is also valid for a maneuvering target. Thus, this proposed guidance law has the advantage of true proportional guidance law, i.e., the missile curvature command decreases to zero as the distance r between missile and target decreases to zero for a nonmaneuvering target. Besides, because the command is applied in a direction normal to the velocity vector of the missile, the speed of the missile can be kept constant, as in pure proportional navigation guidance study.

C. Qualitative Study of Capture Capability of the Designed Guidance Curvature Command

In this section, a qualitative analysis method will be used to determine the sufficient initial condition such that miss cannot occur in consequent engagements. For (17) to be valid, the missile curvature command k_m must be well defined, that is, the denominator of Eqs. (15) and (16) must have nonzero value during the entire engagement. From this requirement, the missile torsion command τ_m is derived in the next section.

From Eq. (17), for $A > 2$, we have

$$r = 0 \Rightarrow \omega = 0$$

Setting $r = 0$, $r' = 0$ in Eq. (1), we have

$$\boldsymbol{t}_m = m\boldsymbol{t}_t$$

This relation is impossible because both $\boldsymbol{t}_m$ and $\boldsymbol{t}_t$ are unit vectors and $m < 1$. Therefore, there are two possibilities at the end of an engagement: 1) $r' = 0$, $r \neq 0$, which means a miss, and 2) $r = \omega = 0$, which means capture.

In the following study, two cases are discussed for three-dimensional engagements: case 1, where k_t and $\omega_0 = 0$, and case 2, where k_t is arbitrary and $\omega_0 \neq 0$. Negative initial closing velocity, i.e., $r'_0 < 0$, is assumed in all of the following discussion.

In case 1, the target flies straight without maneuver and the initial LOSR is zero. From Eq. (15) or (16) with $k_t = 0$ plus Eq. (17), we have $\omega = 0$ and $k_m = 0$. From Eq. (6) with $k_m = k_t = \omega = 0$, we have $r'' = 0$. Because the initial closing velocity $r'_0 < 0$, $r' = r'_0 < 0$ and the distance between missile and target decreases continuously. Therefore, capture will be guaranteed.

In case 2, we have maneuvering target and initial heading error. In the following paragraphs, based on the LOSR response (17), and the characteristics at the end of an engagement, the sufficient initial conditions that guarantee capture for arbitrary target maneuver will be determined. The missile torsion command that supports the argument of capture will be derived in the next section.

From the LOSR response (17), which is based on the designed missile curvature command (15) or (16), we have

$$\omega_0 > 0 \Rightarrow r\omega > 0 \tag{18}$$

in an engagement. From the kinematics relation (3), we have

$$r\omega = m\boldsymbol{t}_t \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) - \boldsymbol{t}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) > 0 \tag{19}$$

Because $r > 0$, $\omega > 0$, $r' < 0$, and $\omega' < 0$ before the end of an engagement, which can be either capture ($r = 0$) or a miss ($r' = 0$, $r \neq 0$), taking the derivative of Eq. (19) with respect to s gives

$$\frac{d}{ds}(r\omega) = \frac{d}{ds}(m\boldsymbol{t}_t \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)) - \frac{d}{ds}(\boldsymbol{t}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r))$$
$$= r'\omega + r\omega' < 0 \tag{20}$$

From Eq. (12), Eqs. (19) and (20) have the following forms:

$$\frac{d}{ds}(r\omega) = \frac{d}{ds}(\boldsymbol{t}_{mp} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)) - \frac{d}{ds}(\boldsymbol{t}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)) < 0 \tag{21}$$

$$r\omega = \boldsymbol{t}_{mp} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) - \boldsymbol{t}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) > 0 \tag{22}$$

The unit tangent vector $\boldsymbol{t}_{me}$ is defined as the fictitious missile missing velocity direction such that, if the missile velocity vector coincides with it, miss occurs. From Eq. (2), we have

$$m\boldsymbol{t}_t \cdot \boldsymbol{e}_r = \boldsymbol{t}_{me} \cdot \boldsymbol{e}_r \tag{23}$$

In three-dimensional engagements, $\boldsymbol{t}_{me}$ belongs to a set of vectors for a specific $\boldsymbol{t}_t$. Define the set Q of $\boldsymbol{t}_{me}$ for a $\boldsymbol{t}_t$ as

$$Q = \{\boldsymbol{t}_{me} \mid m\boldsymbol{t}_t \cdot \boldsymbol{e}_r = \boldsymbol{t}_{me} \cdot \boldsymbol{e}_r\} \tag{24}$$

Setting $\boldsymbol{t}_m = \boldsymbol{t}_{me}$ and $r' = 0$ in Eq. (3), we have

$$(r\omega)_{me} = m\boldsymbol{t}_t \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) - \boldsymbol{t}_{me} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) > 0 \tag{25}$$

Substituting relation (12) into the last equation (25), we have

$$(r\omega)_{me} = \boldsymbol{t}_{mp} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) - \boldsymbol{t}_{me} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) > 0 \tag{26}$$

Next, the minimum value of Eq. (25) or (26) at miss condition will be determined. In three-dimensional engagements, the LOSR at miss is not fixed for a specific $\boldsymbol{t}_t$. It also depends on the direction of the missile velocity vector at miss, which belongs to a cone with missile's position at the apex. Here, $\boldsymbol{e}_r$ is the symmetric axis of this cone, and the missile velocity vector at miss forms the cone surface with fixed semiangle with the symmetric axis as defined in Eq. (24).

The minimum value of Eq. (25) can be obtained as follows. From Eq. (1), we know

$$m\boldsymbol{t}_t - \boldsymbol{t}_{me} = (r\omega)_{me}(\boldsymbol{e}_\omega \times \boldsymbol{e}_r) \tag{27}$$

Because $(r\omega)_{me} > 0, m\boldsymbol{t}_t - \boldsymbol{t}_{me}$ and $\boldsymbol{e}_\omega \times \boldsymbol{e}_r$ must be in the same direction. From Eq. (27), we have

$$((m\boldsymbol{t}_t - \boldsymbol{t}_{me}) \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r))_{\min} = |m\boldsymbol{t}_t - \boldsymbol{t}_{me}|_{\min} \tag{28}$$

where $\boldsymbol{t}_t$ and $\boldsymbol{t}_{me}$ are unit vectors and $m = \text{const} < 1$. From geometry, we know the minimum magnitude of the difference between two vectors happens when both vectors are along the same direction. Thus, in Eq. (28), the minimum value of $1 - m$ will occur when $\boldsymbol{t}_t$ is parallel to $\boldsymbol{t}_{me}$ and points in the same, i.e., $-(\boldsymbol{e}_\omega \times \boldsymbol{e}_r)$, direction. Therefore, the minimum value of Eq. (25) or (26) is

$$((r\omega)_{me})_{\min} = (\boldsymbol{t}_{mp} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) - \boldsymbol{t}_{me} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r))_{\min} = 1 - m > 0 \tag{29}$$

If $\boldsymbol{t}_m = \boldsymbol{t}_{mp}$, based on the definition of $\boldsymbol{t}_{mp}$ [Eq. (12)] and LOSR (17), we have

$$\omega = 0 \Rightarrow r = 0$$

which means capture. If $\boldsymbol{t}_m \in Q$ is as defined in Eq. (24) and from the definition of $\boldsymbol{t}_{me}$ [Eq. (23)], we have

$$r' = 0, \qquad r \neq 0$$

which means miss. Thus, if we have the initial condition

$$0 < r_0\omega_0 = \boldsymbol{t}_{mp_0} \cdot \left(\boldsymbol{e}_{\omega_0} \times \boldsymbol{e}_{r_0}\right) - \boldsymbol{t}_{m_0} \cdot \left(\boldsymbol{e}_{\omega_0} \times \boldsymbol{e}_{r_0}\right) < 1 - m \tag{30}$$

then, due to Eq. (21), $r\omega$ will decrease continuously and always be less than $1 - m$. From Eq. (29), $r\omega$ can never equal to $(r\omega)_{me}$ during the engagement and $\boldsymbol{t}_m$ can never belong to the set Q in Eq. (24). Therefore, miss will not occur during the engagement and capture will occur when $\boldsymbol{t}_m = \boldsymbol{t}_{mp}$.

D. Derivation of Missile Guidance Torsion Commands

The conclusion stated in the last section is guaranteed if the resulting LOSR (17), based on the designed missile curvature command (15) or (16), is valid. But for the LOSR (17) to be valid, we must have the missile curvature command (15) or (16) well defined, i.e., the denominator of the missile curvature command cannot equal to zero during the engagement. If we have the denominator of k_m equal to zero during the engagement, that is,

$$\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) = 0 \tag{31}$$

then from Eq. (5) we have

$$r\omega' + 2r'\omega = m^2 k_t \boldsymbol{n}_t \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)$$

Therefore, if $\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) = 0$ during the engagement, the rate of change of the LOSR, ω', will depend on the target curvature command k_t and the desired LOSR response (17) will no longer be valid. In this case, Eq. (21) may not be valid, as to the capture conclusion just stated. The objective of this section is to design missile torsion command τ_m that when applied, will rotate the $\boldsymbol{n}_m$ such that $\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) \neq 0$ during the engagement and k_m will be kept well defined. Assuming we have the following initial condition:

$$\boldsymbol{n}_{m_0} \cdot \left(\boldsymbol{e}_{\omega_0} \times \boldsymbol{e}_{r_0}\right) = a \neq 0 \tag{32}$$

then $\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)$ must keep the same sign during the engagement. Otherwise, due to continuity of $\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)$, if it changes sign during the engagement, there must exist an instant such that $\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) = 0$. Then, k_m will not be well defined as required. The first method that can keep the sign of $\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)$ unchanged is to keep $\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)$ fixed at its initial value a. Taking the derivative of Eq. (32) with respect to s and applying the Frenet formula, we have

$$\tau_m = k_m \frac{\boldsymbol{t}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)}{\boldsymbol{b}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} - \frac{\boldsymbol{n}_m \cdot (\boldsymbol{e}'_\omega \times \boldsymbol{e}_r)}{\boldsymbol{b}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} + \omega \frac{\boldsymbol{n}_m \cdot \boldsymbol{e}_r}{\boldsymbol{b}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)} \tag{33}$$

For the missile torsion command to be well defined, we must have $\boldsymbol{b}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r$ not equal to zero. Therefore, for k_m and τ_m to be well defined, both Eq. (32) and the following relation must be satisfied in this approach:

$$\boldsymbol{b}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) \neq 0 \tag{34}$$

Equation (32) can be met by introducing the missile torsion command (33). For Eq. (34) to be satisfied, we can modify the initial condition (30). Assuming we have initial condition

$$0 < r_0\omega_0 = \boldsymbol{t}_{mp_0} \cdot \left(\boldsymbol{e}_{\omega_0} \times \boldsymbol{e}_{r_0}\right) - \boldsymbol{t}_{m_0} \cdot \left(\boldsymbol{e}_{\omega_0} \times \boldsymbol{e}_{r_0}\right) < b - m \tag{35}$$

where

$$m < b < 1$$

from Eq. (12), we have

$$-m \leq (\boldsymbol{t}_{mp} \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)) \leq m$$

From the last equation and Eqs. (21) and (35), the possible range of $\boldsymbol{t}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r)$ is as follows:

$$-b < \boldsymbol{t}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r) < m \tag{36}$$

If we have the initial condition (35) and

$$a^2 + b^2 < 1 \tag{37}$$

then due to

$$(\boldsymbol{b}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r))^2 = 1 - (\boldsymbol{t}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r))^2 - (\boldsymbol{n}_m \cdot (\boldsymbol{e}_\omega \times \boldsymbol{e}_r))^2 \tag{38}$$

and Eqs. (32), (36), and (37), we have

$$(\boldsymbol{b}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r)^2 > 1 - a^2 - b^2 > 0 \Rightarrow \boldsymbol{b}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r \neq 0 \tag{39}$$

Therefore, if the initial conditions are Eq. (32) and

$$0 < r_0\omega_0 = \boldsymbol{t}_{mp_0} \cdot \left(\boldsymbol{e}_{\omega_0} \times \boldsymbol{e}_{r_0}\right) - \boldsymbol{t}_{m_0} \cdot \left(\boldsymbol{e}_{\omega_0} \times \boldsymbol{e}_{r_0}\right) = c < 1 - m \tag{40}$$

such that

$$(c + m)^2 + a^2 < 1 \tag{41}$$

and assuming the missile follows the designed torsion command (33), then Eqs. (32) and (34) will be satisfied during engagement and the missile curvature command k_m and torsion command τ_m are both well defined. Then, we have the desired LOSR response (17) and the conclusion in last section. But, in this case, we have the more conservative sufficient initial conditions (32), (40), and (41) instead of condition (30). The second method, instead of keeping $\boldsymbol{n}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r$ equal to a nonzero constant, aims at keeping $\boldsymbol{b}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r$ fixed, i.e.,

$$\boldsymbol{b}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r = \boldsymbol{b}_{m_0} \cdot \boldsymbol{e}_{\omega_0} \times \boldsymbol{e}_{r_0} = d \tag{42}$$

Taking the derivative of Eq. (42) and applying the Frenet formula, we have

$$\tau_m = \frac{\boldsymbol{b}_m \cdot \boldsymbol{e}'_\omega \times \boldsymbol{e}_r}{\boldsymbol{n}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r} - \omega \frac{\boldsymbol{b}_m \cdot \boldsymbol{e}_r}{\boldsymbol{n}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r} \tag{43}$$

If we have initial conditions (40) and (42) such that

$$(c + m)^2 + d^2 < 1 \tag{44}$$

then, due to Eq. (38), we have

$$\boldsymbol{n}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r \neq 0$$

during engagement. From Eq. (15) or (16) and Eq. (43), both k_m and τ_m are well defined. In this case, the more conservative sufficient initial conditions to validate the capture conclusion are Eqs. (40), (42), and (44).

III. Simulation Results

For three-dimensional engagements, computer simulation results are presented, which include varying maneuvering target and an engagement in which the target maneuvering commands include a torsion command. The common initial conditions and constants for all of the following simulations are specified and listed with meter chosen as the physical unit of distance. Miss distance is defined as the closest distance between missile and target before divergence, i.e., $r' > 0$, occurs and the simulation stops. The simulation software would also stop when r drops below 1 m. 1) Initial distance is $r_0 = 10{,}000$ m. 2) Target/missile velocity ratio is $m = 0.4$ ($V_t = 400$ m/s, $V_m = 1000$ m/s). 3) Missile curvature command proportional constant is $A = 3$. 4) Initial position of the missile is $\boldsymbol{r}_{m_0} = (0, 0, 0)$. 5) Initial position of the target is $\boldsymbol{r}_{t_0} = (0, 0, 10{,}000)$. 6) Initial unit tangent vector of the missile is $\boldsymbol{t}_{m_0} = (0.5, 0, 0.866)$. 7) Initial unit normal vector of the missile is $\boldsymbol{n}_{m_0} = (0, 1, 0)$. 8) Initial unit binormal vector of the missile is $\boldsymbol{b}_{m_0} = (-0.866, 0, 0.5)$. 9) Initial unit tangent vector of the target is $\boldsymbol{t}_{t_0} = (0.939, -0.342, 0)$. 10) Initial unit normal vector of the target is $\boldsymbol{n}_{t_0} = (0.342, 0.939, 0)$. 11) Initial unit binormal vector of the target is $\boldsymbol{b}_{t_0} = (0, 0, 1)$. 12) In all three-dimensional engagement simulation cases, the initial condition for $r\omega$ is

$$r_0\omega_0 \cong 0.18 < 1 - m = 0.6$$

The initial engagement configuration is shown in Fig. 2. The simulation cases are listed as follows.

1) Target curvature command is $k_t = 0.000645$ for $r \geq 5000$ m and $k_t = -0.000645$ for $r < 5000$ m, whereas torsion command is $\tau_t = 0$. The simulation results are presented in Figs. 3–6.

2) Target curvature command is $k_t = 0$ for $r \geq 3000$ m and $k_t = -0.000645$ for $r < 3000$ m, whereas torsion command is $\tau_t = 0$ for $r \geq 3000$ m and $\tau_t = 0.007$ for $r < 3000$ m. In this case, the target follows the roll and pull maneuver commonly seen in real world encounters. The simulation results are presented in Figs. 7–10.

In the three-dimensional engagements just listed, Eq. (33) is applied to compute the missile torsion command with $\boldsymbol{n}_m \cdot \boldsymbol{e}_\omega \times \boldsymbol{e}_r$ held

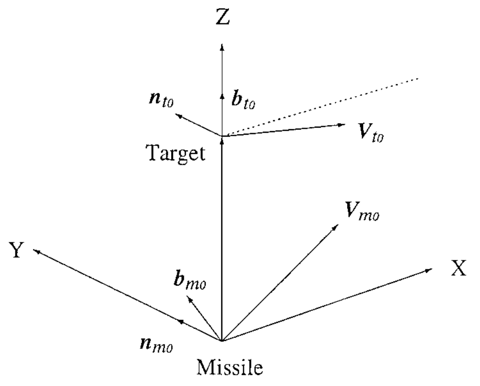

Fig. 2 Initial engagement configuration for three-dimensional engagements.

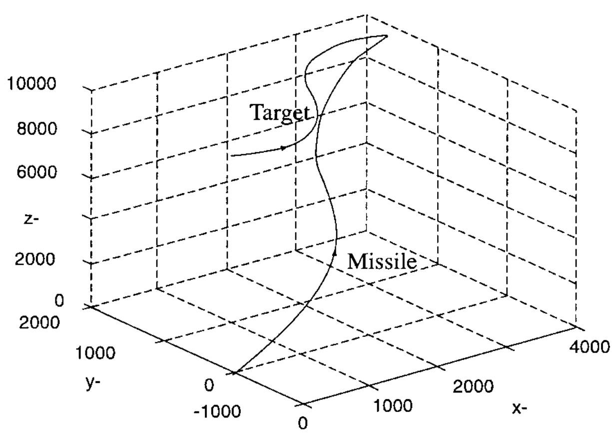

Fig. 3 Case 1, three-dimensional engagement (meter-meter-meter).

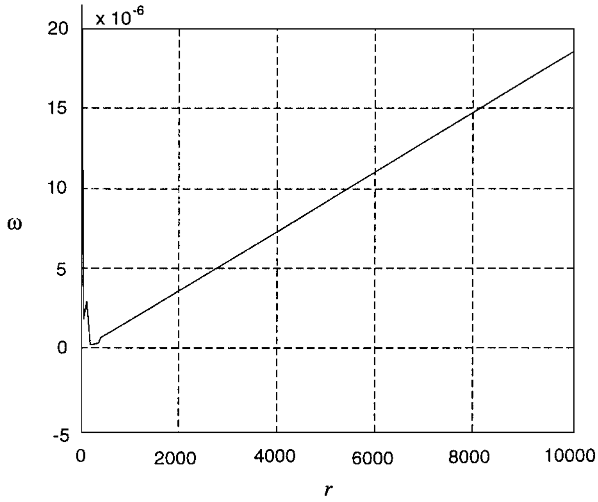

Fig. 4 Case 1, LOSR (1/meter) vs distance (meter) plot.

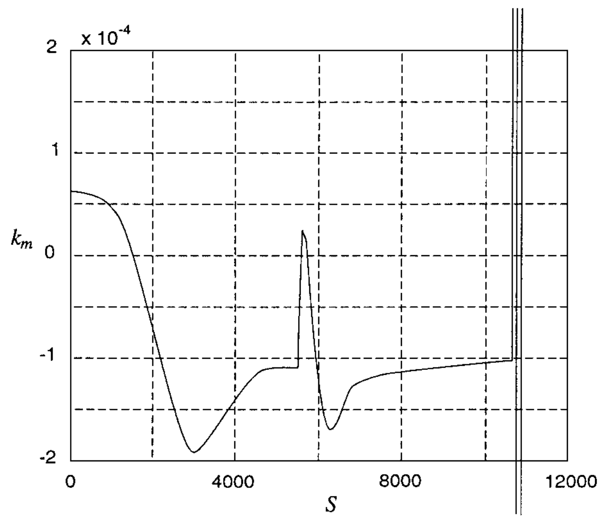

Fig. 5 Case 1, missile's curvature command (1/meter) vs trajectory arc length (meter).

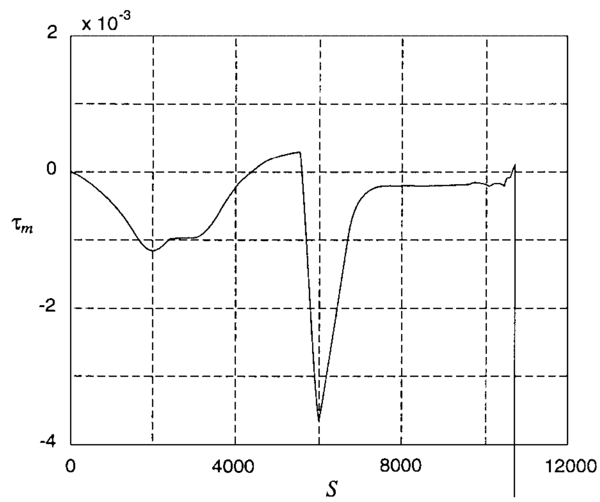

Fig. 6 Case 1, missile's torsion command (1/meter) vs trajectory arc length (meter).

constant at approximately 0.7398. The simulation results for three-dimensional engagements with missile and target as point masses indicate capture and the final missile-target distance are less than 1 m, as presented in Figs. 3 and 7. Traditional proportional guidance command, i.e., Eq. (15) or (16) with its first term removed, fails to achieve capture in both cases. In the following simulation, first-order missile curvature and torsion dynamic responses and time delay in target maneuvering measurement are included. The missile

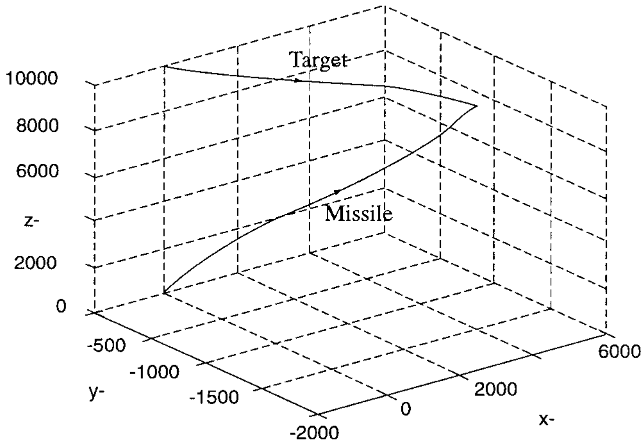

Fig. 7 Case 2, three-dimensional engagement (meter-meter-meter).

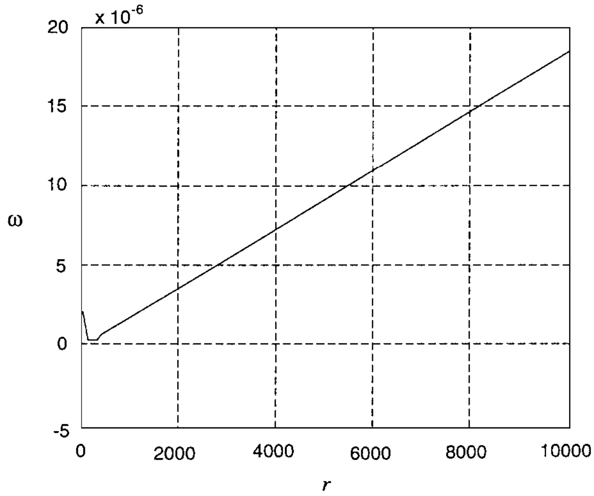

Fig. 8 Case 2, LOSR (1/meter) vs distance (meter) plot.

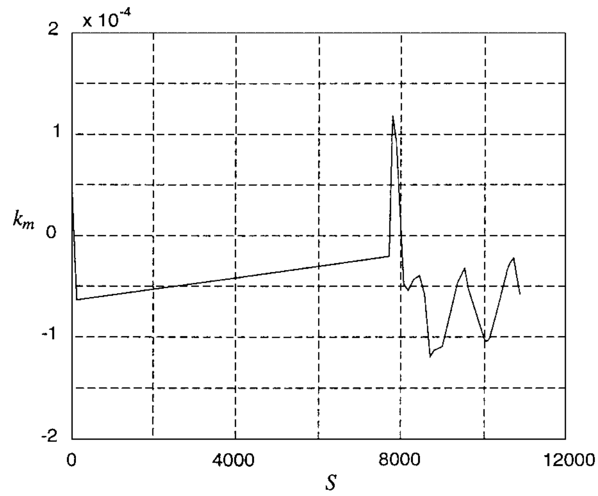

Fig. 9 Case 2, missile's curvature command (1/meter) vs trajectory arc length (meter).

curvature and torsion dynamic responses are assumed to follow the following forms:

$$\frac{k_m}{k_{mc}} = \frac{0.001}{s + 0.001} \qquad \frac{\tau_m}{\tau_{mc}} = \frac{0.005}{s + 0.005}$$

The time delay in target curvature command measurement is assumed to follow the following form:

$$\frac{k_{te}}{k_t} = \frac{0.005}{s + 0.005}$$

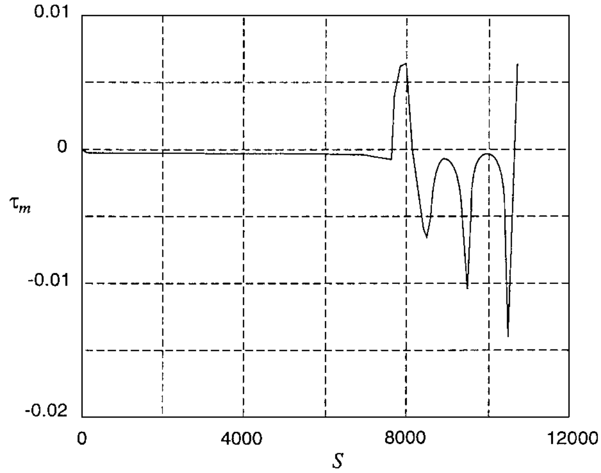

Fig. 10 Case 2, missile's torsion command (1/meter) vs trajectory arc length, meter.

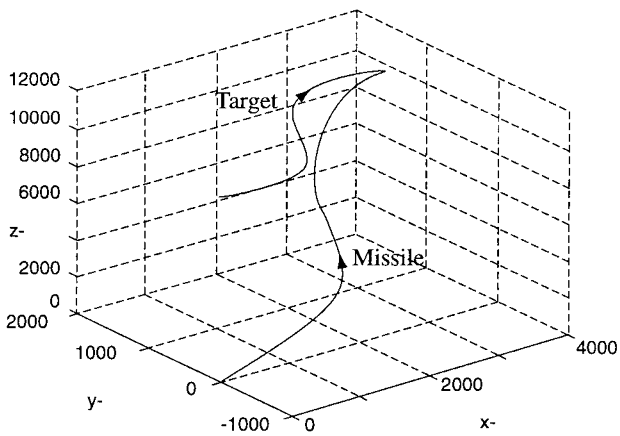

Fig. 11 Case 3, three dimensional engagement (meter-meter-meter).

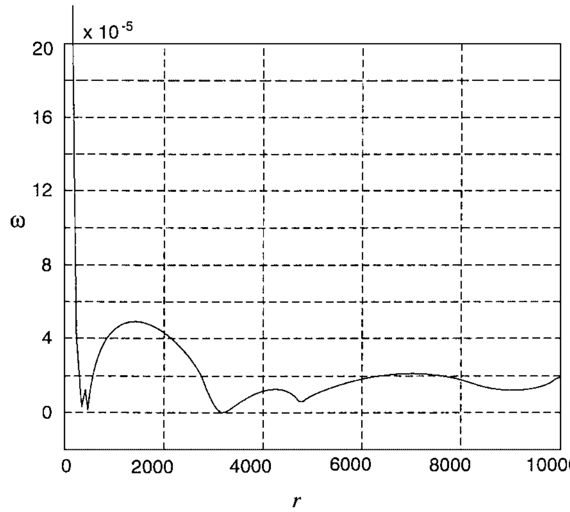

Fig. 12 Case 3, LOSR (1/meter) vs distance (meter) plot.

The target maneuvering commands for case 3 are as follows.

3) Target curvature command is $k_t = 0.000645$ for $r \geq 5000$ m and $k_t = -0.000645$ for $r < 5000$ m, whereas torsion command is $\tau_t = 0$. The simulation results are presented in Figs. 11 and 12.

The miss distance in case 3 is approximately 6.37 m, which also indicates capture result. In Fig. 12, except near capture, the ω vs r plot is well below the $r\omega = 1 - m = 0.6$ curve. Above this curve, it is possible for miss to occur. A summary of results that guarantee capture is presented in Fig. 13.

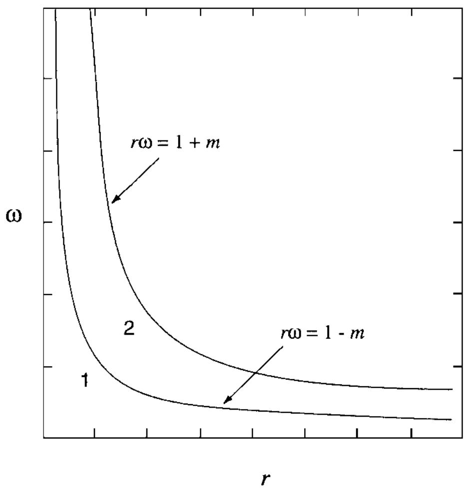

Fig. 13 LOSR (ω) vs distance (r).

IV. Conclusions

The results presented for three-dimensional engagements that guarantee capture for an arbitrary target's maneuvers can be summarized in the ω vs r plot in Fig. 13. With the proposed missile guidance commands, if the engagement starts in region 1, i.e., the initial engagement satisfies the stated sufficient initial condition, due to the characteristics of negative rate of change of $r\omega$ as the engagement proceeds, the engagement will never enter region 2, where miss may occur. Therefore, from the study of possibilities at the end of an engagement, both r and ω will approach zero at capture. According to the separated regions of capture and miss in Fig. 13, capture conclusion in three-dimensional engagements is completely determined by the spatial variation of $r\omega$. Therefore, for any other guidance commands that have the same characteristics of $r\omega$ response, that is, spatial derivative with respect to arc length s along missile trajectory is less than zero, we can have a similar conclusion, and capture will be guaranteed. In this case, we have the same definition of miss condition but capture may occur at nonzero ω because Eq. (17) may not hold. For robustness characteristics of other kinds of guidance commands, if the magnitude of their tangential kinematics relation is always less than $1 - m$, then miss cannot occur. This comment is still valid even if the dynamics of the missile control system is included.

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